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SATELLITE NAVIGATION BY TERRESTRIAL OCCULTATIONS OF STARS I:

GENERAL CONSIDERATIONS NEGLECTING ATMOSPHERIC REFRACTION AND EXTINCTION

ALI M. NAQVI

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AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AFB, OHIO

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FOREWORD

This study was undertaken as part of an investigation of navigation within the solar system by optical means. The objective of the investigation is to evaluate the suitability of various physical phenomena as sources of navigational information and to estimate the accuracy of navigational information obtained by various techniques. The work was supported under Contract No. AF 33(616)-7413 by the Navigational and Guidance Laboratory, Wright Air Development Division, Air Research and Development Command, United States Air Force.

The present report is one of a series of three entitled:

SATELLITE NAVIGATION BY TERRESTRIAL OCCULTATIONS OF STARS

I. General Considerations Neglecting Atmospheric Refraction and Extinction by Ali M. Naqvi (Geophysics Corporation of America Technical Report 62-18-A, October 1962).

II. Considerations Relating to Refraction and Extinction by Ali M. Naqvi (Geophysics Corporation of America Technical Report 62-21-A, October 1962).

III. Interference Due to Brightness of Earth's Atmosphere by Rollin C. Jones and Ali M. Naqvi (Geophysics Corporation of America Technical Report 62-22-A, October 1962).

ABSTRACT

The condition for terrestrial occultation of a star as seen from a satellite is derived in terms of the orbital elements of the satellite and the equatorial coordinates of the star. An occultation equation relating the time of occultation to the orbital elements is also derived. Relevant stellar properties are discussed. The frequency of occultations as a function of perigee distance and eccentricity of the orbit is obtained. A general discussion of the minimum number of stars required for satellite navigation, and the corresponding limiting magnitude of the satellite-borne telescope is given. Except for references to other reports of this series where atmospheric effects are considered, all results are derived on the assumption that the earth is a perfect sphere and has no atmosphere.

SECTION 1

INTRODUCTION

One of the classical methods for measuring the longitude of a place on the earth's surface is based upon the observations of star occultations by the moon (See Chauvenet, 1891, Chap. 10). The method assumes a knowledge of the lunar orbit. Alternatively if the longitude of the observer could be determined by some other method, the star occultations could be used to determine the lunar orbit, or rather to apply small corrections to an assumed orbit. The similarity between this procedure and a proposed method for determining the orbit of an earth-satellite, based upon the measurements of the time of occultations of stars by the earth, is obvious.

A parallel beam of light from a star falls on the earth, which casts a cylinderical shadow whose axis is along the line joining the star and the center of the earth. As seen from the satellite an occultation of this star by the earth will occur, if and only if, some portion of the satellite's orbit passes through the above mentioned cylinderical shadow.

The earth is, in fact, not a perfect sphere and therefore its shadow is not a perfect cylinder. However, correction for known departures from sphericity can be applied. To the extent that there are bound to be some errors in the measurements of the shape of the earth, the exact shape of the shadow would also be uncertain. Then there are mountains on the earth's surface some of which extend to almost 10 km above the earth's surface. Unless their exact locations and height are known, some small errors can arise due to these irregularities.

However, the presence of clouds in the earth's atmosphere, which can completely absorb the star light, and of which the height may vary in a totally unpredictable manner, makes the observations of occultations of stars by the earth's solid surface unreliable for position and orbit measurements. In the second report of this series (Naqvi, 1962; hereafter referred to as Report II), it is shown that criteria other than a geometrical occultation behind the opaque earth can be used for satellite navigation. In this report we shall assume that the earth is a perfect sphere with no atmosphere (and no clouds). Section 2 defines the various orbital elements. In Section 3 a condition for the occultation of a given star by a satellite of a given orbit is derived. Usually an approximate orbit of the satellite is predetermined. This condition is then useful since it allows us to know in advance of the launching of the satellite exactly which stars, (and how many), would be occulted. In Section 4, the occultation equation which relates the time of occultation to the orbital elements is derived. A resumé of relevant stellar properties is given in Section 5, and finally, the expected frequency of star occultations and the limiting magnitudes of the stars to be observed are discussed in Section 6.

In Report II of this series, the effects of refraction and extinction are investigated and in Report III a brief discussion of the interference due to the brightness of the earth's atmosphere is given.

SECTION 2

THE ORBITAL ELEMENTS

In this section we define the six fundamental orbital elements and a few others related parameters. For more complete details the reader may consult any standard text on celestial mechanics or orbit theory (for example Moulton, 1914). For an elliptical orbit six orbital elements and time completely define the position of a body in space. (Figure 1a). These are:

a - semi-major axis, which defines the size of the orbit.

e - eccentricity, which defines the shape or oblateness of the orbit.

T - time of perihelion passage which defines the position of the body in its orbit at a given time.

i - inclination of the orbital plane to the fundamental plane (xy plane in Figure 1a). Note that in the usual orbit theory the ecliptic is taken to be the fundamental plane, but in considering the satellite orbits, the earth's equatorial plane is found more convenient.

Ω - longitude (if the ecliptic coordinate system is used) or right ascension (if the equatorial coordinate system is used) of the ascending node. The ascending node is defined as the point of intersection of the orbit and the fundamental plane, when the body is moving from negative to positive Z direction and i and Ω together define the orientation of the orbital plane with reference to a given coordinate system.

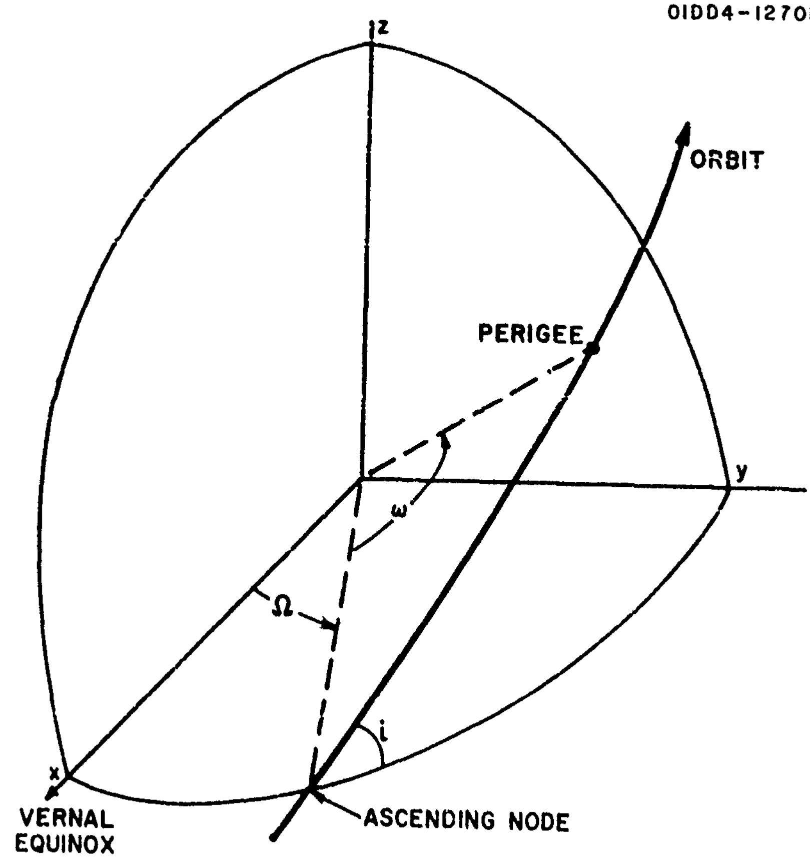

Figure 1a. The orbital elements in the equatorial coordinate system, (xy-plane is the plane of the earth's equator).

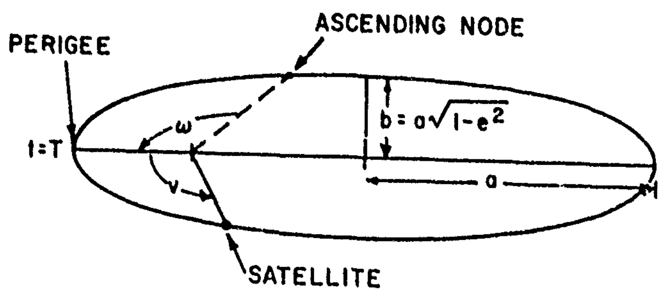

Figure 1b. The orbital elements in the orbital plane.

ω - longitude of the perigee measured from the ascending node along the orbital plane, which defines the orientation of the orbit in its plane.

There are several other orbital parameters which are functions of the above mentioned elements, and which can therefore be used equally well in their place. We shall have occasion to use two such parameters, which are:

P - period of the body which is related to the semi-major axis a in accordance with Kepler's third law;

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G(m + m')} \simeq \frac{4\pi^2}{Gm} \simeq \text{constant} \qquad (2.1)$$

The approximation in Equation (2.1) holds provided the mass of the smaller body m' is negligible compared to the mass of the larger body m. G is the gravitational constant. A convenient way to use Kepler's law is to write it as follows:

$$P = P_o \left({}^{a}/{a_o}\right)^{3/2} \qquad (2.2)$$

where P_o is the period of an imaginary satellite whose semi-major axis is a_o. If a_o = 6378 km, the equatorial radius of the earth, we have

$$P_o = 5069 \text{ sec} = 84.48 \text{ min} = 1.408 \text{ hours} \qquad (2.3)$$

v - true anomaly is the polar angle (Figure 1b) measured from the perigee in the forward direction of motion along the orbit. It is evident that v is independent of the elements which define the orientation of the orbital plane in space, or the orbital orientation in this plane, namely i, Ω, and ω, and is a function of a (or P), e, T and the time t. For derivation of the following results a reference may be made to Moulton

(Chap. 5, 1914).

$$\tan \frac{v}{2} = \left(\frac{1 + e}{1 - e}\right)^{\frac{1}{2}} \tan \frac{E}{2} \tag{2.4}$$

$$E - e \sin E = M \quad \text{(known as Kepler's equation)} \tag{2.5}$$

$$M = 2\pi \left(\frac{t - T}{P}\right) \tag{2.6}$$

When the eccentricity e is small (as is the case for planetary orbits, and for some satellite orbits) a series expansion in e can be obtained

$$\begin{aligned} v = {} & M + 2e \sin M + \frac{5}{4} e^2 \sin 2M \\ & + \frac{e^3}{12} (13 \sin 3M - 3 \sin M) \\ & + \frac{e^4}{96} (103 \sin 4M - 44 \sin 2M) \\ & + \frac{e^5}{960} (1097 \sin 5M - 645 \sin 3M + 50 \sin M) \\ & + \frac{e^6}{960} (1223 \sin 6M - 902 \sin 4M + 85 \sin 2M) \\ & + \cdots \end{aligned} \tag{2.7}$$

The polar equation of an ellipse in its simplest form is

$$r = \frac{a(1 - e^2)}{1 + e \cos v} \tag{2.8}$$

SECTION 3

CONDITIONS FOR OCCULTATION

It can be seen from elementary geometrical consideration that if the earth-star vector is in the plane of the satellite orbit, an occultation of this star must occur (See Figure 2) during an orbital revolution. On the contrary, if the earth-star vector is perpendicular to the orbital plane an occultation can not occur. The general condition is derived in this section.

There is another way to look at the occultation geometry from the point of view of an observer on the satellite, which is depicted in Figure 3. A cone is drawn just enveloping the earth with the satellite at its apex and the line joining the satellite to the earth's center as its axis. The semi-angle of the "shadow cone" is

$$\beta = \sin^{-1} \frac{a_o}{r} \tag{3.1}$$

where a_o is the earth's radius and r, the distance from the earth's center to the satellite. r is sometimes expressed in terms of h, the height above the earth's surface, as

$$r = a_o + h \tag{3.2}$$

A star whose direction, as viewed from the satellite, lies on the forward half of the instantaneous surface of the cone is said to be at the point of ingress, whereas a star whose direction lies on the other half of the instantaneous surface of the cone is said to be at the point of egress.

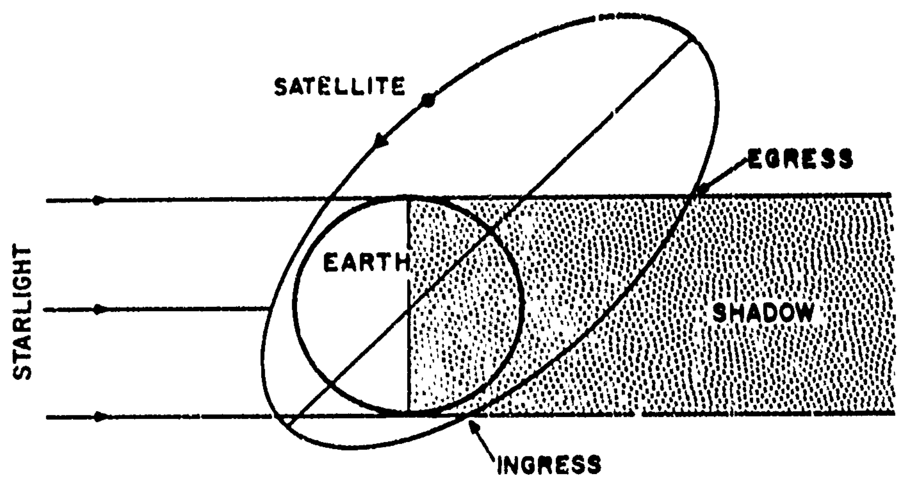

Figure 2. Occultation geometry when the earth-star vector is in the orbital plane of the satellite.

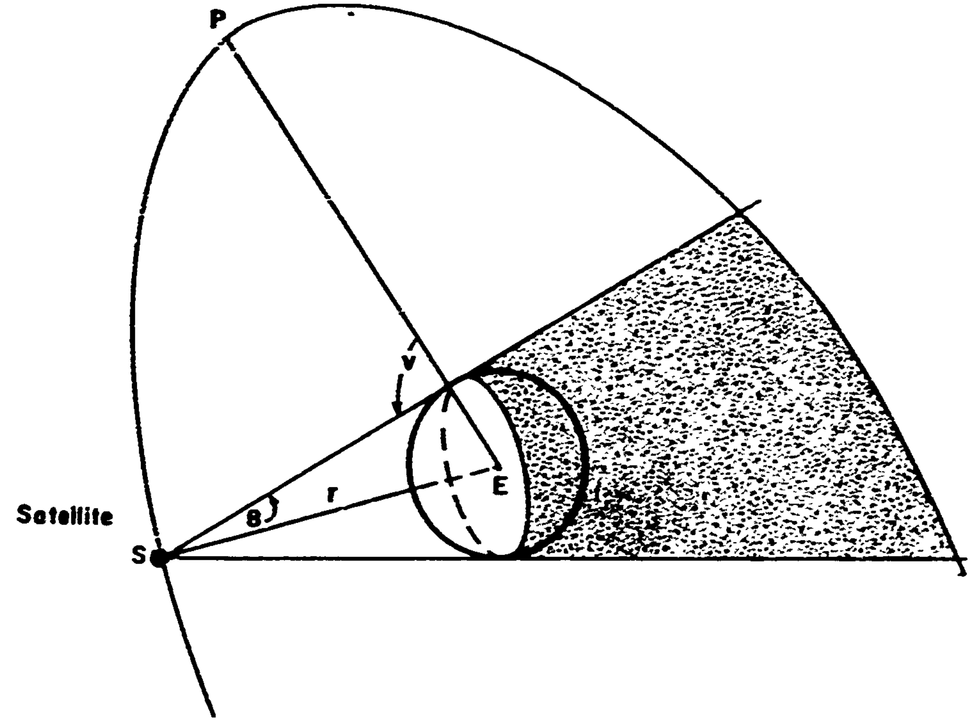

Figure 3. Occultation geometry

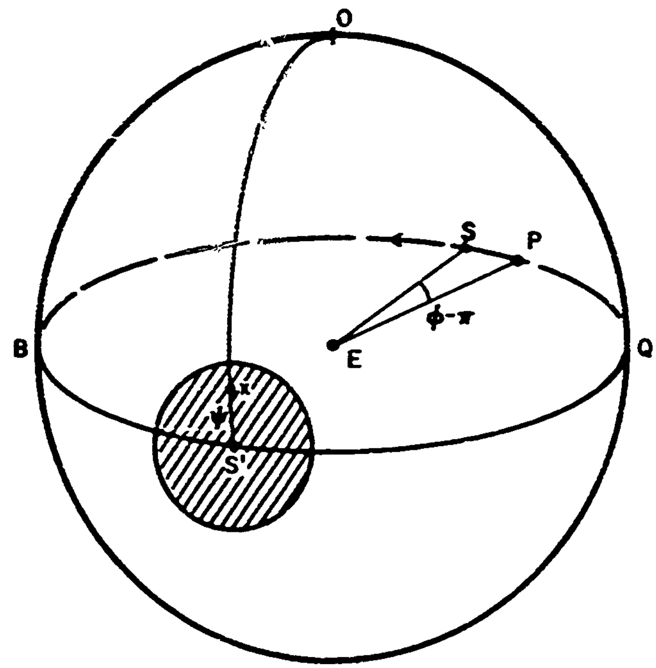

Figure 4 The condition for occultation.

As the satellite revolves around the earth, the "shadow cone", which can be imagined to be attached to the satellite, also revolves. For an elliptical orbit the radius vector r is given by Equation (2.8). Variations in r cause changes in the semi-angle of the shadow cone β, which as a function of the true anomaly v, becomes,

$$\beta = \sin^{-1} \frac{a_o}{a} \frac{1 + e \cos v}{1 - e^2} \tag{3.3}$$

For any orbit the maximum and minimum values of β are

$$\beta(\max) = \sin^{-1} \frac{a_o}{a(1 - e)} \tag{3.4}$$

$$\beta(\min) = \sin^{-1} \frac{a_o}{a(1 + e)} \tag{3.5}$$

These are functions of only two of the orbital coordinates, namely a and e. It is evident that the condition for occultation is that the line joining the star and the satellite must pass through the shadow cone at some point along the satellite orbit.

Consider a celestial sphere (Figure 4) with the earth at its center and the plane of the satellite's orbit, BSPQS', as its fundamental plane. Let P denote the perigee and S the projection on the celestial sphere of the instantaneous position of the satellite in its orbit. The "shadow cone", described above would intersect the celestial sphere in the shaded circle whose center S' is π radians away from S. The angular radius of this circle is defined by Equation (3.3).

Consider a star X whose latitude is ψ and whose longitude ϕ (measured from perigee P along the orbital plane in the direction of the satellite's motion) is the same as the instantaneous longitude of S'. This system of coordinates will be called the "orbital" coordinate system. The condition that an occultation of this star, as seen by a satellite S (of given orbit), would occur is that

$$\psi \leq \sin^{-1} \frac{a_o(1 - e \cos \phi)}{a(1 - e^2)} \tag{3.6}$$

Since stellar positions are usually given in right- ascension and declination, some relations between these coordinates and the orbital latitude and longitude are required. To derive these relations, consider the celestial sphere drawn in Figure 5, in which the plane of the earth's equator is the fundamental plane. This differs from the usual drawings of orbits of bodies in the solar system which have the ecliptic as the fundamental plane of the celestial sphere.

The following results can be obtained very easily by standard methods of spherical trigonometry.

$$\sin \psi = -\sin i \cos \delta \sin (\alpha - \Omega) + \cos i \sin \delta \tag{3.7}$$

$$\cos \psi \sin (\omega + \phi) = \cos i \cos \delta \sin (\alpha - \Omega) + \sin i \sin \delta \tag{3.8}$$

$$\cos \psi \cos (\omega + \phi) = \cos (\alpha - \Omega) \cos \delta \tag{3.9}$$

Equations (3.8) and (3.9) can be further simplified to yield

$$\cos \psi \sin \phi = \cos i \cos \delta \sin (\alpha - \Omega) \cos \omega + \sin i \sin \delta \cos \omega - \cos (\alpha - \Omega) \cos \delta \sin \omega \tag{3.10}$$

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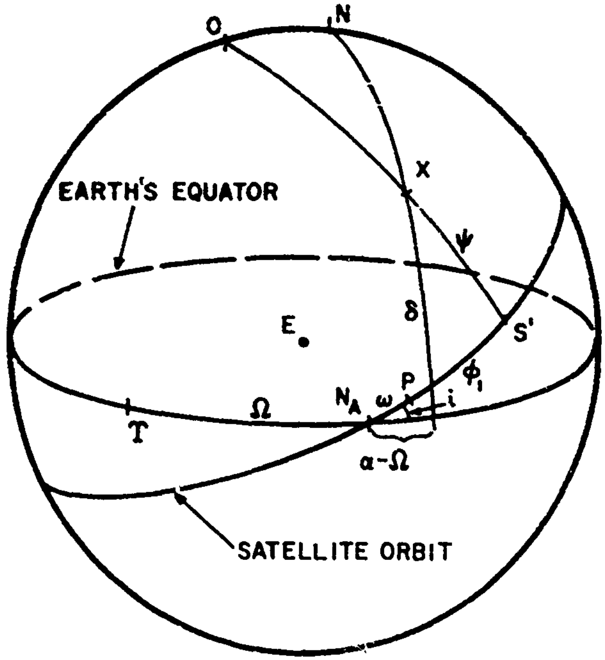

Figure 5. The equatorial and the "orbital" coordinate systems.

$$\cos \psi \cos \phi = \cos i \cos \delta \sin (\alpha - \Omega) \sin \omega + \sin i \sin \delta \sin \omega + \cos (\alpha - \Omega) \cos \delta \cos \omega \tag{3.11}$$

The condition for occultation becomes

$$\frac{a}{a_o} (1 - e^2) (\cos i \sin \delta - \sin (\alpha - \Omega) \sin i \cos \delta) + e \cos \left(\tan^{-1} \frac{\sin i \sin \delta + \sin (\alpha - \Omega) \cos i \cos \delta}{\cos (\alpha - \Omega) \cos \delta} \right) \leq 1 \tag{3.12}$$

SECTION 4

DERIVATION OF ORBITAL ELEMENTS

Consider the celestial sphere drawn in Figure 6. The instantaneous position of the satellite is marked at S and the projection of the earth's "shadow" on the celestial sphere is again depicted by the shaded circle whose center S' is π radians from S. The true anomaly of the satellite v equals PS. The star X, whose orbital latitude XX' = ψ and whose orbital longitude PBX' = ϕ is at the point of ingress.

It is easy to see that XS' (= β) is the instantaneous angular radius of the earth's shadow, given by Equation (3.3), and S'X' = (ϕ - π - v). Now consider the spherical triangle XX'S' whose angle XX'S' is a right angle. Solving this triangle we obtain

$$\cos \beta = \cos \psi \cos (\phi - \pi - v) = - \cos \psi \cos (v - \phi) \tag{4.1}$$

We now use Equations (3.10) and (3.11) to change from the orbital to the equatorial coordinates of the star, and get *

$$- \cos \beta = - \left[1 - \left(\frac{a_o (1 + e \cos v)}{a(1 - e^2)} \right)^2 \right]^{\frac{1}{2}}$$

$$= \cos i \cos \delta \sin (\alpha - \Omega) \sin (v + \omega) + \sin i$$

$$\sin \delta \sin (v + \omega) + \cos \delta \cos (\alpha - \Omega) \cos (v + \omega) \tag{4.2}$$

The observations consist of the measurement of the time of ingress (or egress) of several stars. Six such time measurements will yield six equations from which the six orbital elements can be calculated.

* This equation has also been derived by scientists at Control Data Corporation (Private communication from Robert Lillestrand).

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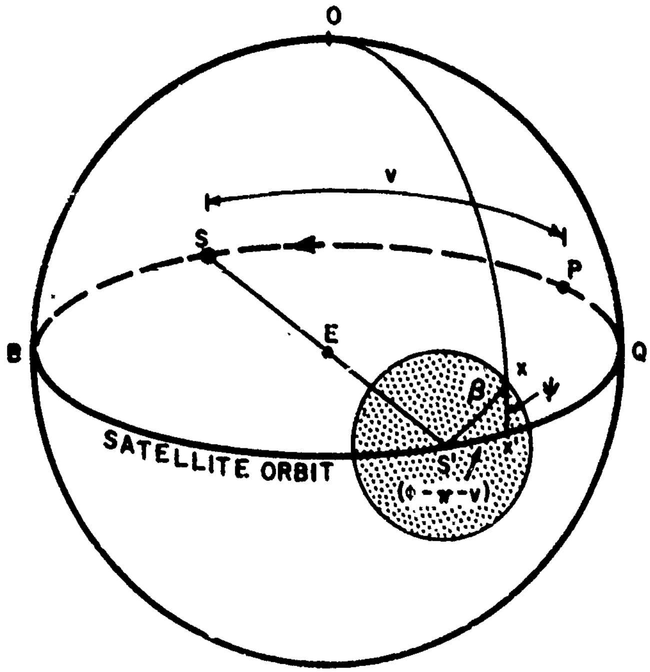

Figure 6. The occultation geometry at the instant of ingress.

SECTION 5

RESUMÉ OF STELLAR PROPERTIES

5.1 SPECTRAL CHARACTERISTICS AND TEMPERATURES

The spectral classification of stars divides them into several groups labelled B, A, F, G, K and M. There are a few other groups also, labelled W, O, Q, R, N and S, but since they consist of only about 1% of all the apparently bright stars, they will not be considered here. Each spectral class is further divided into 10 subclasses, labelled by numbers from zero to 9.

In addition the stars are divided into groups according to their intrinsic (as opposed to apparent) luminosity. The most important luminosity classes are the supergiants, the giants and the main sequence stars, although the following more complete classification is frequently used.

I	Supergiants (including c stars)
II	Bright giants
III	Giants
IV	Sub-giants
V	Main sequence (also called dwarfs)
Sd	Sub-dwarfs
w	White dwarfs

Except for the presence of a few absorption and emission lines, the spectral intensity distribution of a star closely approximates the Planck law. Table 1 (taken from Allen, 1957) gives the effective temperatures of stars of various spectral and luminosity classes. The effective temperature T_e is defined in terms of the luminosity L

TABLE 1

EFFECTIVE TEMPERATURES AND COLOR INDICES FOR VARIOUS SPECTRAL AND LUMINOSITY CLASSES (taken from Allen, 1957)

Sp	C			T_e		
	Main seq.	Giants	Super-giants	Main seq.	Giants	Super-giants
B0	-0.42			22 000		
B5	-0.36			14 000		
A0	-0.17			10 700		
A5	+0.03			8 500		
F0	+0.15	+0.22		7 400		
F5	+0.28	+0.40	+0.34	6 500		
G0	+0.42	+0.60	+0.70	5 900	5 200	4 900
G5	+0.58	+0.82	+1.05	5 500	4 600	4 300
K0	+0.77	+1.06	+1.4	4 900	4 100	3 800
K5	+1.08	+1.35		4 200	3 600	3 300
M0	+1.3	+1.5		3 600	3 400	3 000
M5	+1.5			2 800	2 800	

(total energy output per second) as follows,

$$L = 4\pi R^2 T_e^4 \sigma$$

where R is the radius of the star and σ is the Stefan-Boltzmann constant. There are several different definitions of stellar temperatures used by astrophysicists (color temperature, brightness temperature, etc.) but the effective temperature is physically the most significant, and it is a parameter of great theoretical usefulness (See, for example, Aller 1953, Chap. 6).

From the wide range of stellar temperatures shown in Table 1, it is evident that the intensity distribution of different stars in different spectral regions would vary very much. This is shown in Figure 7 (taken from a recent paper by Ramsey, 1962 *), where the energy received at the top of the earth's atmosphere from some of the brightest stars (of different spectral and luminosity classifications) is plotted against the wave length. Thus it can be seen that a red star which is visually bright, for example Betelgeux, has very little energy at the short wave length region, say below 5000 A.

Some relevant data concerning the stars for which curves of Figure 7 are drawn is listed in Table 2.

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* The following stars are omitted from the figure of Ramsey:

R Hydrae because of its low brightness

α Crucis and Altair because the temperatures used by Ramsey are in error. Their effective surface temperatures are 21,000° K and 8,600° K respectively, whereas Ramsey used 2,81?° K and 7,500° K.

Antares because Ramsey labelled two different curves for this star. (It appears that one of these curves is for Antares and the other for Aldebaran.)

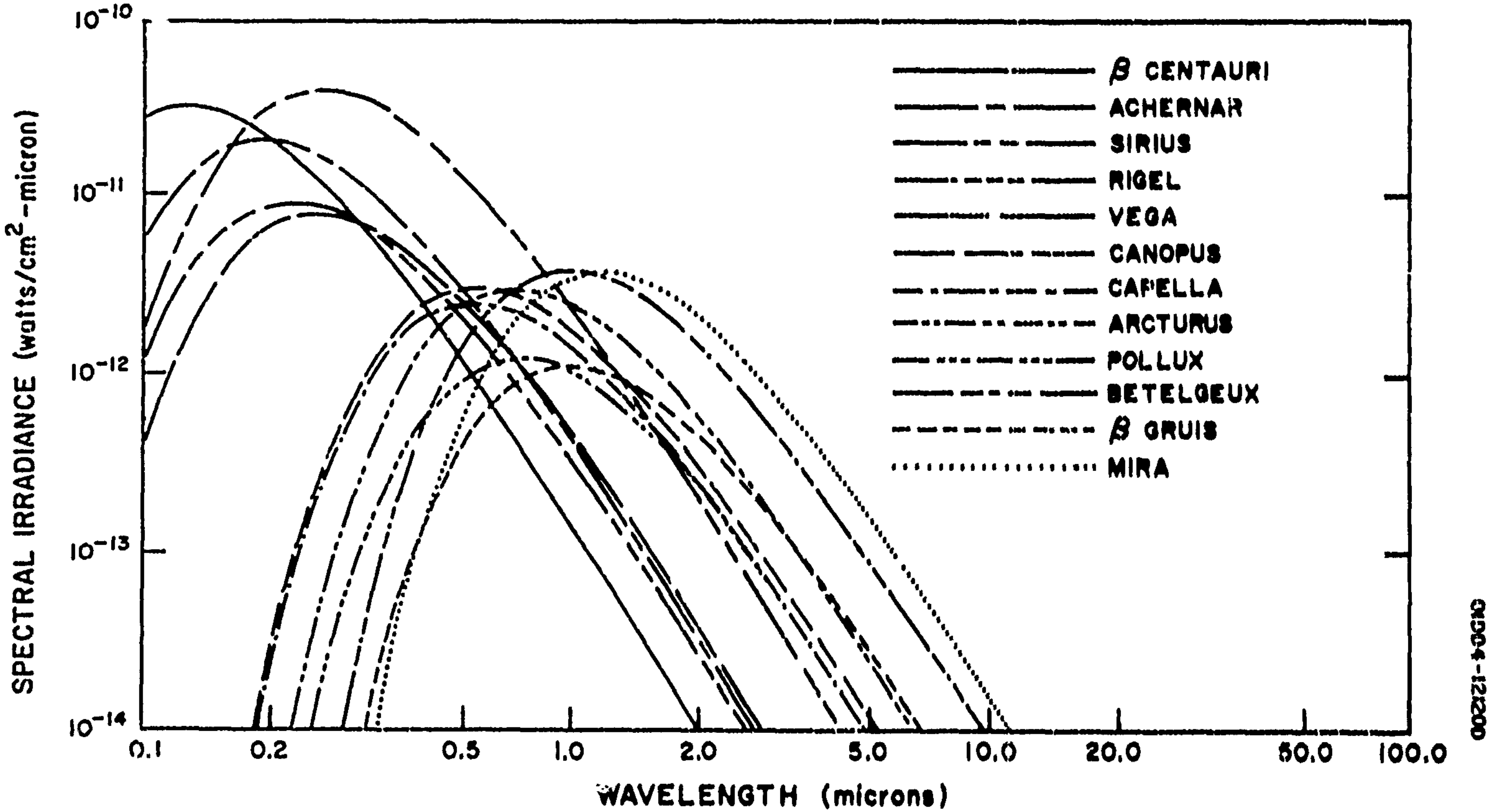


Figure 7. Spectral irradiance of selected bright stars at the top of the atmosphere (Redrawn from Ramsay, 1962).

TABLE 2

MAGNITUDES, TEMPERATURES, SPECTRAL AND LUMINOSITY CLASSES FOR SELECTED STARS

Star	m_v	m_{pg}	T_e(°K)	Sp	Luminosity Class
Sirius	-1.43	-1.58	11 200	A1	V
Achernar	0.55	0.27	15 000	B5	IV
Rigel	0.14	-0.03	12 500	B8	I
Mira (variable)	1.9	3.3	2 390	M6	
Betelgeuse (variable)	0.7	2.44	3 100	M2	I
Vega	0.05	-0.08	11 200	A0	IV
β Gruis	2.14	3.64	2 810	M3	II
Pollux	1.12	2.00	4 200	G9	III
β Centauri	0.69	0.31	21 000	B1	II
Canopus	-0.77	-0.67	8 000	F0	I
Arcturus	0.03	1.24	4 100	K0	III
Capella	0.13	0.87	5 500	G1	III

5.2 INTENSITY CHARACTERISTICS

The range in the apparent brightnesses, or apparent magnitudes of stars (due in part to range in intrinsic brightness and in part to that in distances) is enormously great. The naked eye can, under optimum conditions, see stars of visual magnitude m_v up to 6. Through a telescope of 2" aperture (a likely instrument to be used in a satellite) stars of visual magnitude up to 10 or 12 can be detected.

The wave length region best suited for measurements required in satellite navigation is discussed in Report II. It is approximately the region from 3000 A to 5000 A. On the other hand the visual magnitude refers to stellar brightness in the spectral region of greatest visual sensitivity, which centers at 5560 A and extends from approximately 5000 A to 6200 A.

Assuming Planck's law to hold for stars, a knowledge of the effective surface temperature and visual magnitude can be used to derive total irradiance in any other spectral region (such as that from 3000 to 5000 A). However, for this preliminary discussion this is not considered necessary. Instead we can use the photographic magnitudes as a criterion of brightness in the above-mentioned spectral region, since this region is almost coincident with the region of maximum sensitivity of the photographic plate. The difference between photographic magnitude m_{pg} and the visual magnitude m_v is known as color index, C.I.

$$C.I. = m_{pg} - m_v$$

The color index for individual bright stars is given by Allen (1957, Section 112), and for various spectral and luminosity classes it is given in Table 1. Right ascension, declination, photographic magnitudes and spectral and luminosity classes for the 50 (photographically) brightest stars is given in Table 3 (Arranged in decreasing order of brightness; data taken from Allen 1957, Section 112).

TABLE 3

FIFTY BRIGHTEST STARS (PHOTOGRAPHIC MAGNITUDE)

STAR		α	δ	m_{PG}	Spectral Class	Luminosity Class
Sirius	α C Ma	$6^h40.8^m$	-16°35'	-1.58	A1	V
Canopus	α Car	$6^h21.7^m$	-52°38'	-0.67	F0	I
Vega	α Lyr	$18^h33.6^m$	+38°41'	-0.08	A0	V
Rigel	β Ori	$5^h\ 9.7^m$	- 8°19'	-0.03	B8	I
Achernar	α Eri	$1^h34.0^m$	-57°45'	0.27	B5	IV
	β Cen	$13^h56.8^m$	-59°53'	0.31	B1	II
	α Cru	$12^h21.0^m$	-62°33'	0.40	B1	IV
	α Cen	$14^h32.8^m$	-60°25'	0.43	G2, K3	
Procyon	α C Mi	$7^h34.1^m$	+ 5°29'	0.66	F5	IV
Spica	α Virgo	$13^h19.9^m$	-10°38	0.68	B1	V
Altair	α Aql	$19^h45.9^m$	+ 8°36'	0.87	A7	V
Capella	α Aur	$5^h\ 9.3^m$	+45°54'	0.87	G1	III
	β Cru	$12^h41.9^m$	-59° 9'	0.94	B0	III
Regulus	α Leo	$10^h\ 3.0^m$	+12°27'	1.10	B8	V
Fomalhaut	α Ps A	$22^h52.1^m$	-30° 9'	1.13	A3	V
	ϵ C Ma	$6^h54.7^m$	-28°50'	1.20	B2	II
Deneb	α Cyg	$20^h18.6^m$	+44°55'	1.20	A2	I
Arcturus	α Boö	$14^h11.1^m$	+19°42'	1.24	K0	III
	λ Sco	$17^h26.8^m$	-37° 2'	1.27	B2	IV
	γ Ori	$5^h19.8^m$	+ 6°16'	1.37	B2	III

TABLE 3 (Continued)

STAR		α	δ	m_{pg}	Spectral Class	Luminosity Class
Castor	α Gem	$7^h 28.2^m$	+32° 6'	1.39	A1	V
	ε Ori	$5^h 31.1^m$	- 1°16'	1.43	B0	I
	β Tau	$5^h 20.0^m$	+28°32'	1.46	B8	III
	ε U Ma	$12^h 49.6^m$	+56°30'	1.47	A1p	
	γ Vel	$8^h 6.4^m$	-47° 3'	1.51	WC7	
	η U Ma	$13^h 43.6^m$	+49°49'	1.56	B3	V
	α Gru	$22^h 1.9^m$	-47°27'	1.58	B5	V
	β Car	$9^h 12.1^m$	-69°18'	1.59	A0	III
	ζ Ori	$5^h 35.7^m$	- 2° 0'	1.60	09	I
	β C Ma	$6^h 18.3^m$	-17°54'	1.63	B1	III
	α Pav	$20^h 17.7^m$	-57° 3'	1.65	B3	IV
	ε Sgr	$18^h 17.5^m$	-34°26'	1.75	B9	IV
	α Sgn	$18^h 49.1^m$	-26°25'	1.77	B3	V
	δ Vel	$18^h 41.9^m$	-54°21'	1.83	A0	V
	γ Gem	$6^h 31.9^m$	+16°29'	1.85	A1	IV
	κ Ori	$5^h 43.0^m$	- 9°42'	1.85	B0	I
	ζ Pup	$8^h 0.1^m$	-39°43'	1.86	05	
	α And	$0^h 3.2^m$	+28°32'	1.91.	B8p	III
	δ Ori	$5^h 26.9^m$	- 0°22'	1.91	09	II
	β Aur	$5^h 52.2^m$	+44°56'	1.93	A2	IV
Algol	β Per	$3^h 1.6^m$	+40°34'	1.95	B8	V
Pollux	β Gem	$7^h 39.2^m$	+28°16'	2.00	G9	III

TABLE 3 (Continued)

STAR	α	δ	m_{pg}	Spectral Class	Luminosity Class
κ Sco	$17^h35.6^m$	-38°59'	2.00	B2	IV
γ Cas	$0^h50.7^m$	+60°11'	2.01	B0e	IV
ϵ Cen	$13^h33.6^m$	-52°57'	2.06	B1	IV
γ Cen	$12^h36.0^m$	-48°35'	2.08	A0	IV
α Cr B	$15^h30.4^m$	+27° 3'	2.10	A0	V
β Leo	$11^h44.0^m$	+15° 8'	2.12	A3	V
η Cen	$14^h29.2^m$	-41°43'	2.13	B3p, A2p	
ζ U Ma	$13^h19.9^m$	+55°27'	2.14	A2	V

5.3 FREQUENCY OF STARS OF DIFFERENT APPARENT MAGNITUDES

Figure 8 gives the number of stars, Ω_m, brighter than a given photographic or visual magnitude (m_{pg} or m_v). The increase of Ω_m with m is approximately logarithmic, but tapers off for higher magnitudes. It is also evident from this figure that the number of stars brighter than a given photographic magnitude is less than two-thirds the number brighter than the same visual magnitude, and this ratio decreases as we go towards fainter stars, becoming less than one-half at magnitude 10. (See Table 4.)

5.4 DISTRIBUTION OF STARS OVER THE CELESTIAL SPHERE

Figure 9 shows the distribution of the fifty photographically brightest stars using equatorial coordinates. It is evident that the distribution is far from uniform. The two solid lines represent parallels of galactic latitude (+15° and -15°). Thirty-two of the fifty stars are within this narrow band.

The non-uniformity of distribution increases rapidly as we go towards greater and greater limiting magnitudes, as is shown by the plots in Figures 10 and 11 where log $N_{m_{pg}}$ (number of stars brighter than photographic magnitude m_{pg}, per square degree) is plotted against galactic latitude (for m_{pg} = 5, 10, 15 and 20) and against m_{pg} (for galactic latitudes ±0°, ±30°, ±60° and ±90°) respectively. It is evident that for any limiting photographic magnitude m_{pg}, there are fewer stars at high galactic latitudes than near the galactic equator, and that this ratio decreases with increasing limiting magnitude.

The model of our galaxy as a lens-shaped system with the sun located in an eccentric position near the galactic plane explains the observed concentration of stars in low galactic latitudes.

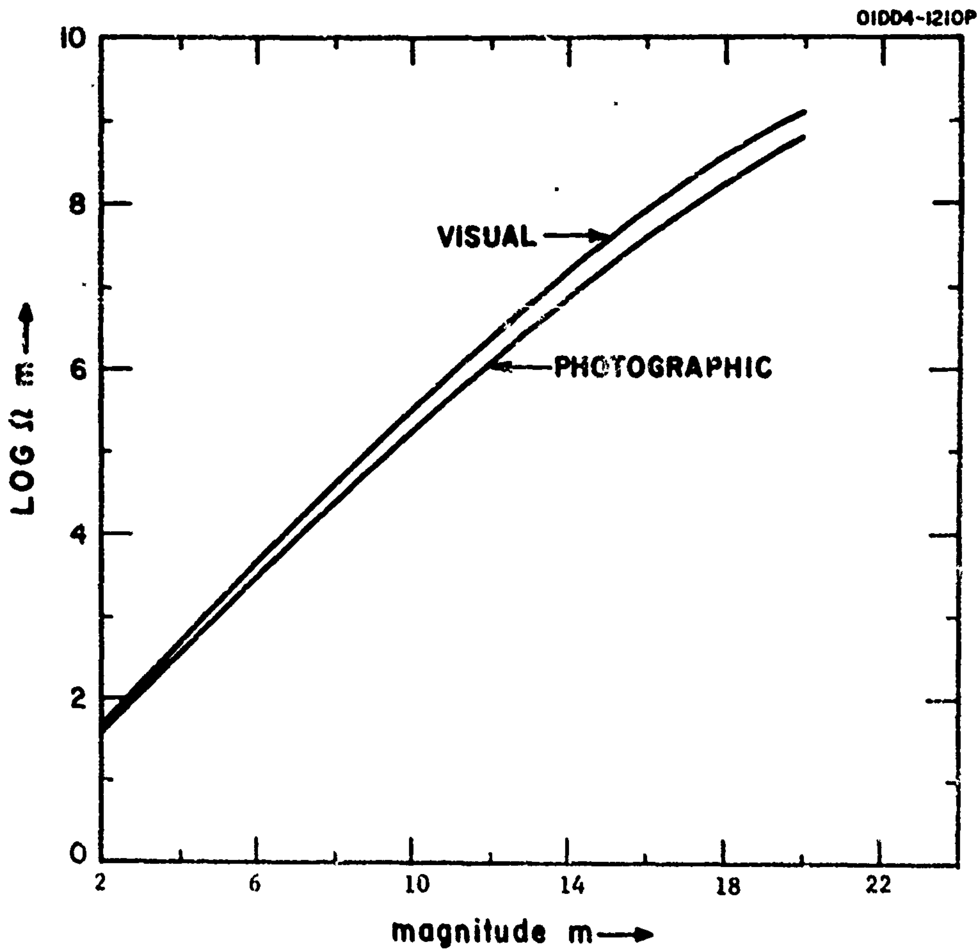


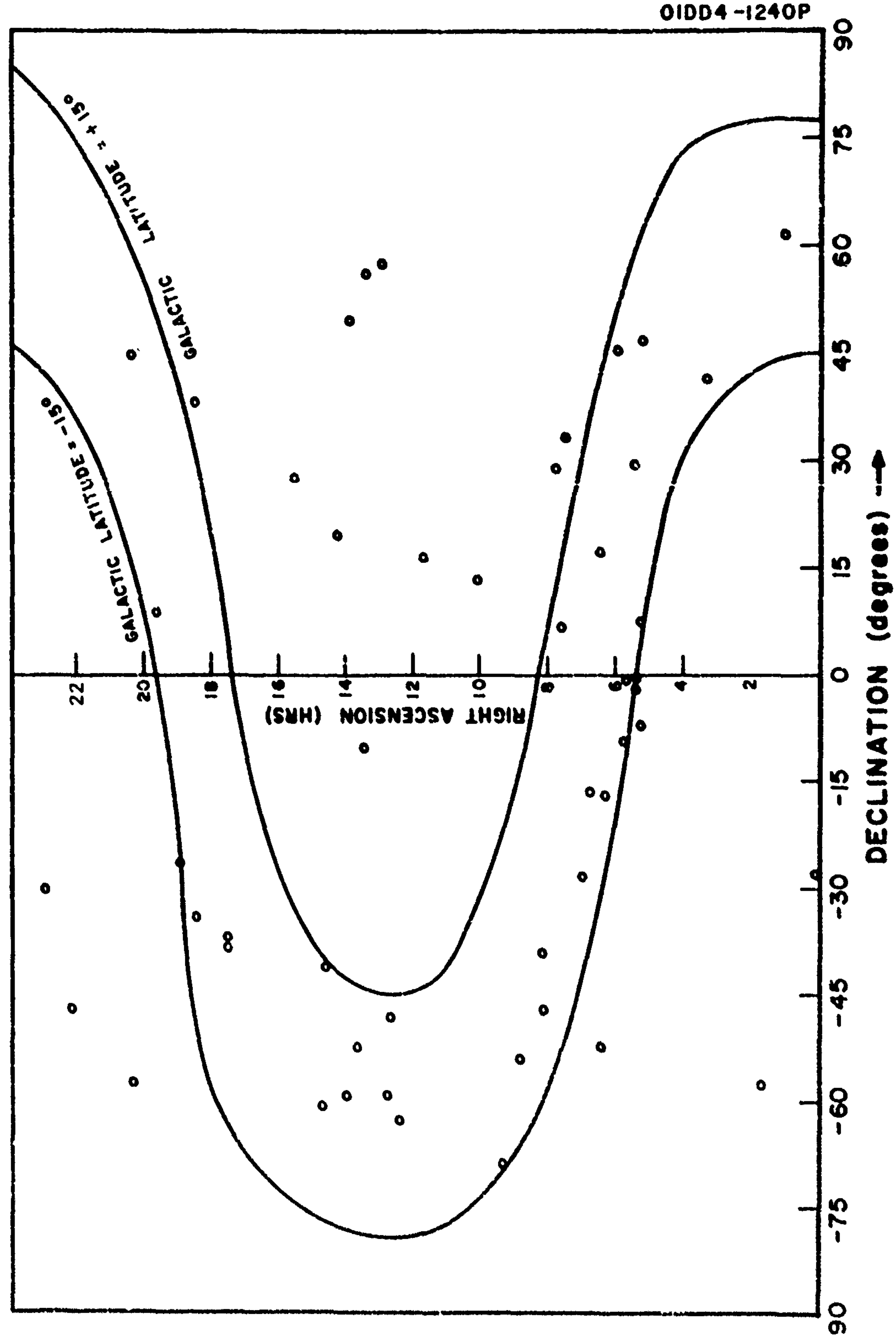
Figure 9. The number of stars Ω_m brighter than apparent magnitude m.

TABLE 4

NUMBER OF STARS BRIGHTER THAN APPARENT MAGNITUDE m

(VISUAL AND PHOTOGRAPHIC)

m	Ω_m (photographic)	Ω_m (visual)
4	3.236×10^2	5.370×10^2
6	3.020×10^3	4.898×10^3
8	2.344×10^4	4.169×10^4
10	1.738×10^5	3.467×10^5
12	1.202×10^6	2.455×10^6
14	7.586×10^6	1.514×10^7
16	3.981×10^7	8.318×10^7
18	1.820×10^8	3.802×10^8
20	6.166×10^8	1.175×10^9



OIDD4-1240P
GALACTIC LATITUDE = +15°
GALACTIC LATITUDE = -15°
RIGHT ASCENSION (HRS)
22
20
18
16
14
12
10
8
6
4
2
DECLINATION (degrees) →
90
75
60
45
30
15
0
-15
-30
-45
-60
-75
-90

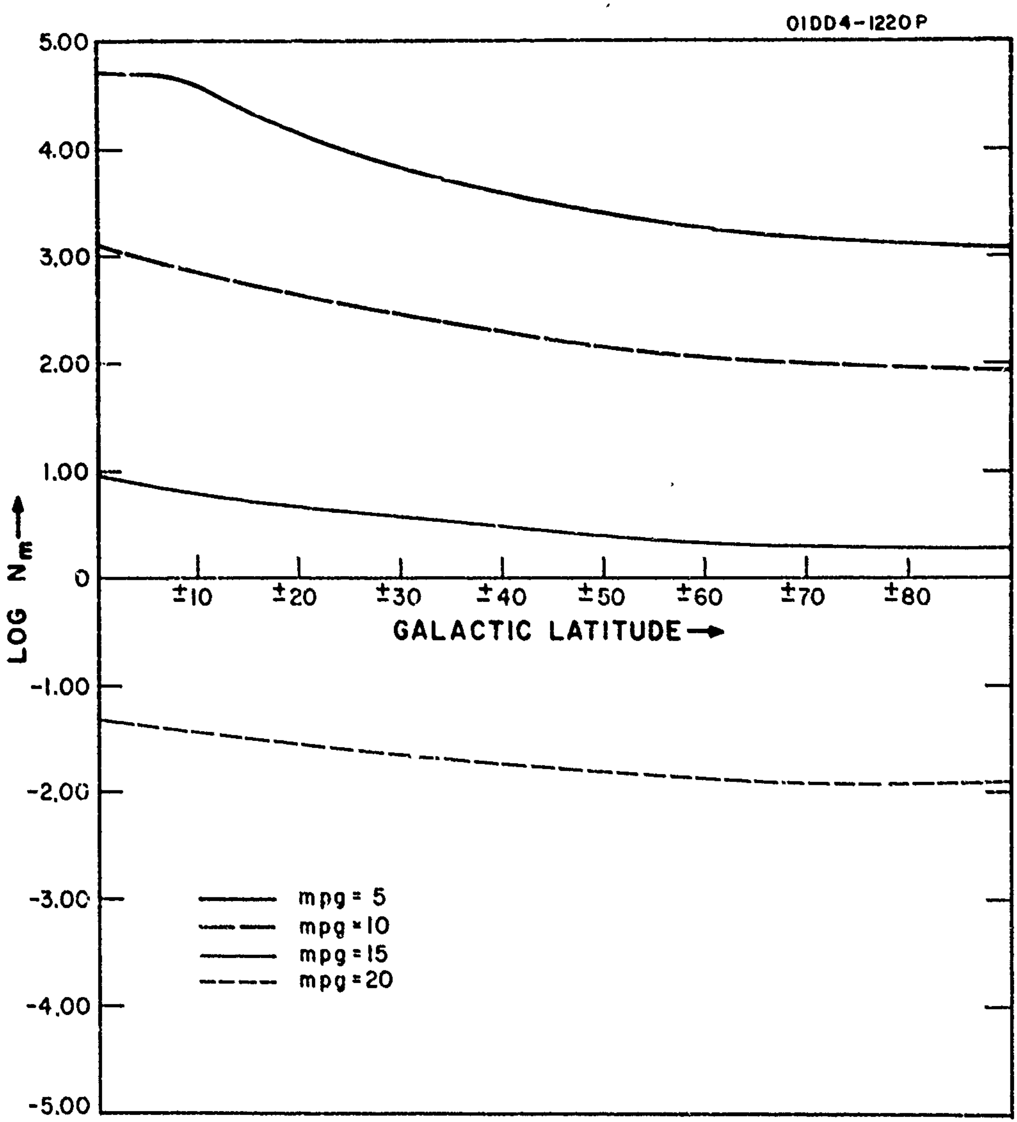


Figure 10. Number of stars brighter than magnitude m_{pg} at various galactic latitudes.

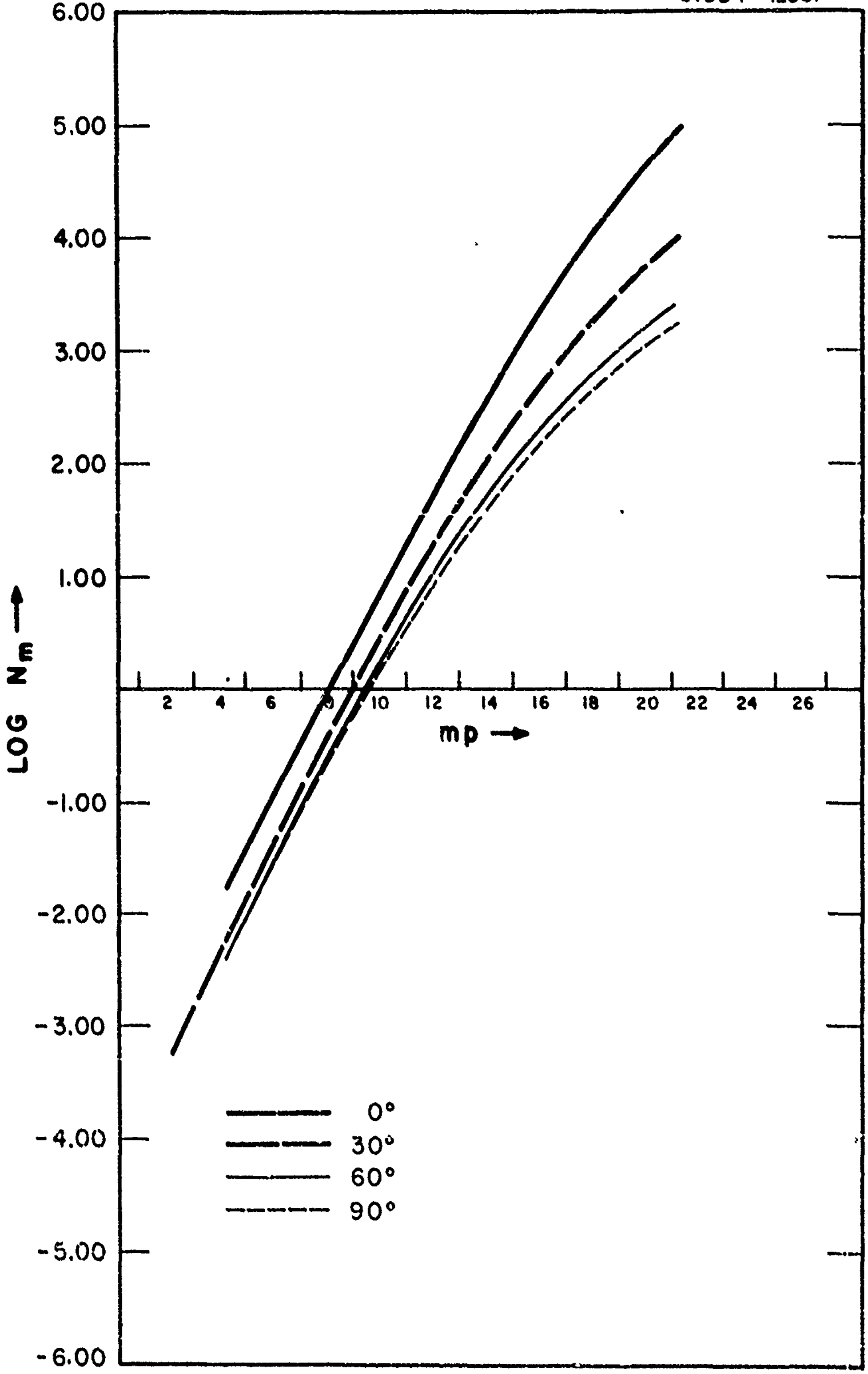


Figure 11. Number of stars brighter than magnitude m_{pg} for various values of m_{pg}

SECTION 6

FREQUENCY OF STAR OCCULTATIONS

6.1 THE FRACTIONAL AREA OF A CELESTIAL SPHERE OCCULTED INSTANTANEOUSLY

The semi-angle of the earth's "shadow cone", according to Equation (3.1), is

$$\beta = \sin^{-1} \frac{a_o}{r} .$$

This corresponds to a solid angle

$$\omega = 2\pi(1 - \cos \beta) \qquad (6.1)$$

The fractional area of a celestial sphere occulted

$$= \frac{\omega}{4\pi} = \tfrac{1}{2}(1 - \cos \beta)$$

$$= \tfrac{1}{2}\left(1 - \left[1 - \frac{a_o^2}{r^2}\right]^{\frac{1}{2}}\right)$$

$$= \tfrac{1}{2}\left(1 - \left[1 - \frac{a_o^2}{a^2(1 - e)^2}\right]^{\frac{1}{2}}\right) \quad \text{at the perigee}$$

$$= \tfrac{1}{2}\left(1 - \left[1 - \frac{a_o^2}{a^2(1 + e)^2}\right]^{\frac{1}{2}}\right) \quad \text{at the apogee}$$

6.2 THE FRACTIONAL AREA OF A CELESTIAL SPHERE OCCULTED PER PERIOD

Consider the celestial sphere drawn in Figure 12 where the central plane is that of the satellite orbit. The band of the celestial sphere enclosed between the dotted parallels of latitude denotes the shadow band for a circular orbit. However for an elliptical orbit the band will not have a uniform thickness. The maximum thickness will

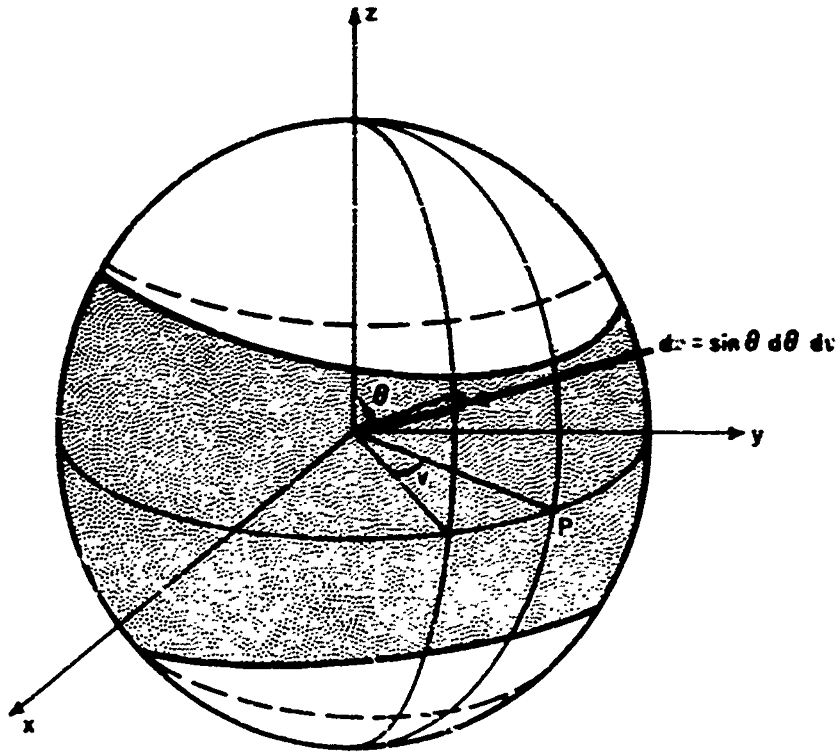


Figure 12. Area of celestial sphere occulted. The thickness of the "shadow band" varies during an orbital revolution for an elliptical orbit.

occur at the apogee point (corresponding to the satellite location at the perigee) and the minimum at the perigee point. The shaded area represents such a shadow band of varying thickness. It is drawn to correspond to an ellipse whose perigee distance equals the radius of the circular orbit whose "shadow cone" is represented by the dotted parallels of latitude. The fractional area occulted per period is given by:

$$F = \frac{1}{4\pi R^2} \int_0^{2\pi} \left[\int_{\frac{\pi}{2} - \beta}^{\frac{\pi}{2} + \beta} R^2 \sin \theta \, d\theta \, dv \right]$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \left[- \cos \theta \right]_{\frac{\pi}{2} - \beta}^{\frac{\pi}{2} + \beta} dv$$

$$= \frac{1}{4\pi} \int_0^{2\pi} 2 \sin \beta dv$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{a_o(1 - e \cos v)}{a(1 - e^2)} \quad dv$$

$$= \frac{1}{2\pi} \cdot \frac{a_o}{a(1 - e^2)} \left[v - e \sin v \right]_o^{2\pi}$$

$$= \frac{1}{2\pi} \cdot \frac{a_o}{a(1 - e^2)} \quad (2\pi)$$

$$= \frac{1}{1 + e} \cdot \frac{a_o}{r_p} \tag{6.3}$$

where $r_p = a(1 - e)$ is the perigee distance. The average fractional area occulted per hour is,

$$= \frac{1}{P}\frac{1}{1 - e^2}\left(\frac{a_o}{a}\right)$$

$$= \frac{1}{P_o}\frac{1}{1 - e^2}\left(\frac{a_o}{a}\right)^{5/2}$$

$$= \frac{1}{P_o}\frac{(1 - e)^{3/2}}{1 + e}\left(\frac{a_o}{r_p}\right)^{5/2} \qquad (6.4)$$

where P_o is given by Equation (2.3).

For a circular orbit of radius r_p, the average fractional area occulted per hour is:

$$= \frac{1}{P_o}\left(\frac{a_o}{r_p}\right)^{5/2} \qquad (6.5)$$

which is $\frac{1 + e}{(1 - e)^{3/2}}$ times as large as for an ellipse whose perigee distance equals the radius of the given circle. For $e = \frac{1}{2}$ this factor equals 4.2

It is to be noted that there can be a fairly substantial variation in the actual rate of occultation of the fractional area of the celestial sphere on account of the fact that the speed of the satellite as well as the instantaneous fractional area occulted, have their maximum values at the perigee and their minimum values at the apogee.

6.3 OCCULTATION FREQUENCY

We saw in Section 4, that the minimum number of occultation observations of distinct stars required to calculate all six orbital elements is six. Since experimental errors are inherent to all measurements, it would be desirable, in order to reduce the probable errors of measurements, to make approximately twelve observations and obtain least square solutions for the unknown elements. This seems particularly desirable in view of the variable and somewhat unknown radiance (due to aurorae, air glow, etc.) of the earth's atmosphere through which the star light would have to pass on its way to the satellite measuring device. This particular problem is discussed in another report of this series (Jones and Naqvi 1962, hereafter referred to as Report III).

We expect that in most cases, the approximate orbital elements would be known even before the launch, and this, in conjunction with the condition of occultation derived in Section 3, would permit us to calculate exactly which (and how many) stars will be available for occultation observations. However, some general results giving the frequency of occultation can be obtained if we assume a uniform distribution of stars over the sky. The fact that the stars are concentrated in low galactic latitudes (See Section 5) will result, depending upon the inclination of the orbital plane, in a higher or lower occultation frequency than that derived on the basis of an assumed uniform distribution of the same number of stars. If the orbital plane is inclined at a small angle to the galactic plane, more occultations would occur than predicted, and vice versa. Since it would be desirable to have twelve occultations even if the orbital plane were at right angles to the galactic plane, it is estimated that a total of approximately 25 star occultations should be allowed for if we assume a uniform distribution of stars. Very approximately it would mean that for certain favorable orbits (low inclinations to the galactic plane) the fulfillment of the conditions would give us fifty or more stars whereas for the least favorable orbits there will be about twelve stars available.

Let us assume that N stars are uniformly distributed over the sky. From Equation (6.3), the number of stars occulted per period of revolution is

$$f_p = \frac{N}{1 + e} \frac{a_o}{r_p} \quad . \tag{6.6}$$

The average number of stars occulted per hour, according to Equation (6.4) is

$$f_h = \frac{N}{P_o} \frac{(1 - e)^{3/2}}{1 + e} \left(\frac{a_o}{r_p}\right)^{5/2} \quad . \tag{6.7}$$

In order that q star occultations can be observed per satellite period, the total number of stars available over the entire sky should be

$$N_q = q(1 + e) \frac{r_p}{a_o} \quad . \tag{6.8}$$

Similarly if we wish to observe an average of q star occultations per hour, the total number of stars over the entire sky should be

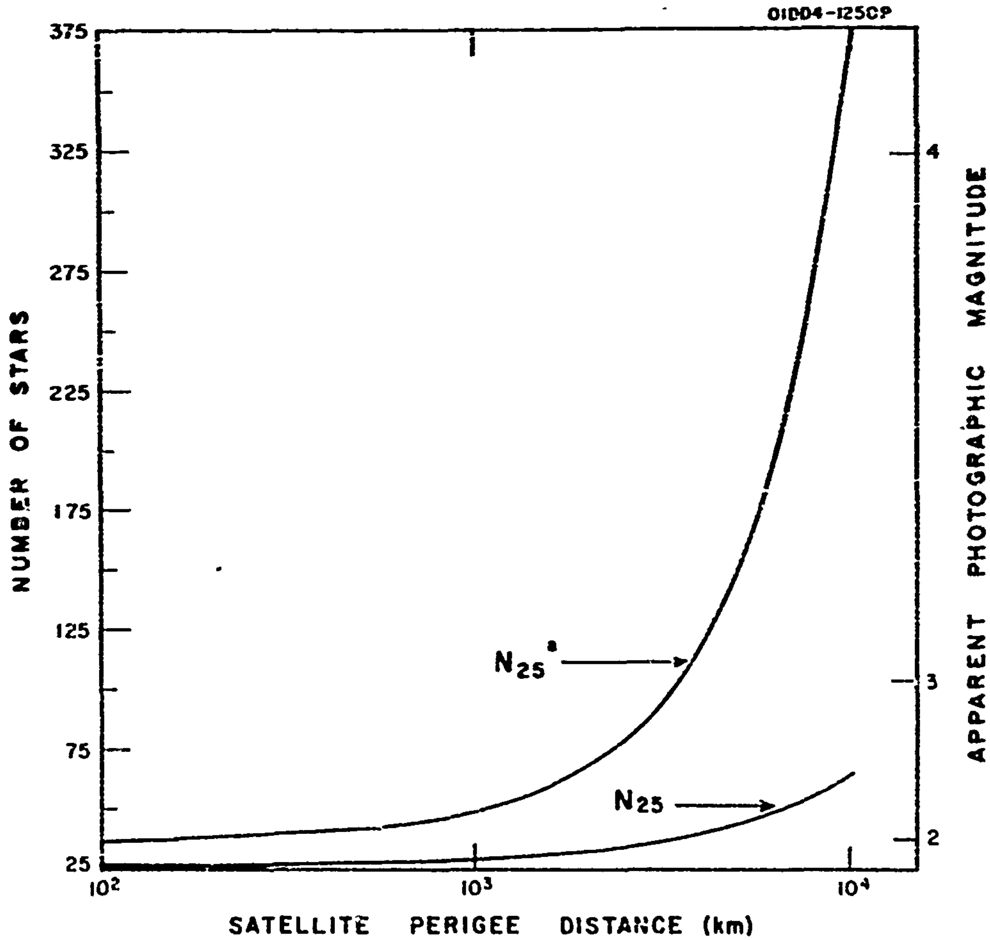


Figure 13. Number of stars required and the corresponding apparent photographic magnitudes for 25 occultations per period (N_{25}) and per hour (N^*_{25}) for a circular orbit ($e = 0$).

$$N_q^* = qP_o \frac{1 + e}{(1 - e)^{3/2}} \left(\frac{r_p}{a_o}\right)^{5/2} \tag{6.9}$$

In Figure 13, we have plotted N_q and N_q^* respectively against $h_p (= r_p - a_o)$ for circular orbits ($e = o$) and for $q = 25$. For orbits of any other eccentricity these values should be multiplied with $(1 + e)$ and $\frac{1 + e}{(1 - e)^{3/2}}$ respectively. The magnitude scale corresponding to the number of stars is given on the right side of the graph.

Although the present report deals primarily with the case neglecting the atmosphere, there are two effects due to the atmosphere, discussed in Reports II and III, which affect the occultation frequency and the number of stars required (or the limiting stellar magnitude), which should be mentioned here. First, due to Rayleigh scattering, the sunlit half of the earth's atmosphere has such a large intensity that no useful observations of stellar occultation against this bright foreground are possible. As a result the frequency of occultations is effectively reduced by $\frac{1}{2}$. The minimum requirement on the number of stars is increased by 2, and the limiting magnitude of the detecting instrument by approximately 0.65 magnitude. Second, due to partial extinction of star light (again due to Rayleigh scattering) the light intensity is decreased by approximately a factor of two, at the instant of observation (See Report II). This does not affect the frequency of occultations, or the requirement on the number of stars, but does increase the limiting apparent magnitude for which the instrument should be designed by approximately 0.75. When the two effects are combined, the requirement on limiting magnitude is increased by 1.4.

TABLE 5

NUMBER OF STARS REQUIRED AND THE CORRESPONDING INSTRUMENTAL LIMITING MAGNITUDE FOR 25 OCCULTATIONS PER PERIOD (N_{25}) AND PER HOUR (N^{*}_{25}) FOR A FEW SELECTED ORBITS

Perigee Height h_p = 200 km

Eccentricity	semi-major axis a (km)	Period (hours)	Number of Stars N_{25}	Limiting Magnitude (photographic)	N^{*}_{25}	Limiting Magnitude (photographic)
0	6578	1.475	26	1.58	38	1.91
.01	6644	1.497	26	1.58	39	1.93
.5	13156	4.171	39	1.93	161	3.38

Perigee Height h_p = 2000 km

Eccentricity	semi-major axis a (km)	Period (hours)	Number of Stars N_{25}	Limiting Magnitude (photographic)	N^{*}_{25}	Limiting Magnitude (photographic)
0	8378	2.106	33	1.83	70	2.55
.01	8463	2.152	33	1.83	72	2.56
.5	16756	5.995	49	2.14	295	3.92

NOTE: (1) Circular and elliptical orbits are considered in this table but only the circular orbits are considered in Figure 13.

(2) The limiting magnitude may be raised by as much as 1.4 magnitudes on account of extinction and radiance of the earth's atmosphere (See end of this section).

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